Physics 606 Exam 2

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work.

An answer, even if correct, will receive zero credit unless it is obtained via the work shown.

Do all your work on blank sheets, and turn them in as scanned files, writing your name clearly.

It is implicit that you pledge to communicate with no one during the exam (except for questions to me about the meaning of the problems).

1. In quantum field theory, we have operators a and $a^\dagger$ which destroy and create particles, and operators b and $b^\dagger$ which destroy and create antiparticles, with a Hamiltonian having the essential form

$$H = \left(a^\dagger a + b^\dagger\, b \pm 1\right) \hbar\omega$$

and the commutation or anticommutation relation

$$\begin{aligned} \left[a, a^\dagger\right]_\mp &\equiv aa^\dagger \mp a^\dagger a &= 1 \\ \left[b, b^\dagger\right]_\mp &\equiv bb^\dagger \mp b^\dagger a &= 1 \end{aligned}$$

where the upper sign holds for bosons and the lower sign for fermions. The other commutators or anticommutators are zero, just as for the harmonic oscillator.

Here these are time-dependent Heisenberg operators $a\,(t)$ etc., and the usual Heisenberg equation of motion holds with a commutator in both cases. (The above relations are all true for one Fourier component $\overrightarrow{k}$ or momentum $\overrightarrow{p} = \hbar\,\overrightarrow{k}$, and one polarization.)

(a) (5) Show that

$$[AB, C] \equiv [AB, C]_- = A\,[B, C]_\mp \pm [A, C]_\mp\, B$$

(b) (10) Obtain the Heisenberg equations of motion for $a\,(t)$ and $b\,(t)$.

(c) (5) Solve the equations to obtain $a\,(t)$ and $b\,(t)$ in terms of ω, $a\,(0)$, and $b\,(0)$.

2. In this problem, the operators and states are given in the coordinate representation.

But if you wish, in doing the problem, you can change an angular momentum operator like $\widehat{L}_x$ to its Hilbert space equivalent L_x, and a state like Ψ or $Y_{\ell m}$ to its Hilbert space equivalent $|\Psi\rangle$ or $|\ell m\rangle$.

I.e., you can work either in the coordinate representation or in the Hilbert space description.

Recall that

$$\widehat{L}_{\pm} Y_{\ell,m} = \sqrt{\ell\,(\ell+1) - m\,(m \pm 1)}\,\hbar\, Y_{\ell,m\pm 1} \qquad , \qquad \widehat{L}_{\pm} = \widehat{L}_x \pm i\widehat{L}_y \; .$$

A system is in a state of angular momentum given by

$$\Psi = aY_{1,1} + bY_{1,0} + cY_{1,-1}$$

where

$$|a|^2 + |b|^2 + |c|^2 = 1 \; .$$

(a) (10) Calculate the expectation value of $\widehat{L}_x$. Give your answer in terms of a, b, and c (and their complex conjugates).

(b) (10) Calculate the expectation value of $\widehat{L}^2$.

(c) (10) If

$$\widehat{L}_x \Psi = \hbar \Psi$$

what are the possible values of a, b, and c?

3. A particle has the Hamiltonian

$$H = \frac{\overrightarrow{p}^2}{2m} + V\left(\overrightarrow{r}\right) \ .$$

(a) (10) Calculate the rate of change of the orbital angular momentum operator $\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$ in terms of $\overrightarrow{r}$ and $\overrightarrow{\nabla} V$.

(b) (10) Now let $\langle A \rangle$ be the expectation value of an operator A. For classical mechanics to give a valid description,we would need

$$\frac{d\left\langle \overrightarrow{L} \right\rangle}{dt} = \langle \overrightarrow{r} \rangle \times F\left(\langle \overrightarrow{r} \rangle\right)$$

where F is the force. Give a clear proof that this either is always true or that it is not always true.

4. At time $t = 0$, the wavefunction for a hydrogen atom is

$$\psi\left(\overrightarrow{r}, 0\right) = \frac{1}{\sqrt{10}}\left(2\,\psi_{1,0,0} + \psi_{2,1,0} + \sqrt{2}\,\psi_{2,1,1} + \sqrt{3}\,\psi_{2,1,-1}\right)$$

where, as usual, $\psi_{n,\ell,m}$ is the normalized wavefunction corresponding to these quantum numbers.

(a) (15) Calculate the expectation value of the energy in eV.

(b) (15) Calculate the probability of finding this atom in the state with $\ell = 1$ and $m = +1$ as a function of the time t, for all n.